Similar technique can be used for functions with radical (i.e., something like $\sqrt{x}$).

Example 2.4.6. Find $\displaystyle\lim_{x\to+\infty}\frac{(3x-1)/x}{(\sqrt{3x^2+1})/x}$

Solution. The term with highest degree of the denominator is x^2. But we need to take square root. So we divide the nominator and the denominator by $\sqrt{x^2}=x$. We have

$$\lim_{x\to+\infty}\frac{3x-1}{\sqrt{3x^2+1}} = \lim_{x\to+\infty}\frac{\frac{1}{x}(3x-1)}{\frac{1}{x}\sqrt{3x^2+1}}$$

$$= \lim_{x\to+\infty}\frac{3-\frac{1}{x}}{\sqrt{3+\frac{1}{x^2}}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

|| quotient rule

$$\frac{\lim_{x\to+\infty}(3-\frac{1}{x})}{\lim_{x\to+\infty}\sqrt{3+\frac{1}{x^2}}} = \frac{\lim_{x\to+\infty}3-\lim_{x\to+\infty}\frac{1}{x}}{\sqrt{\lim_{x\to+\infty}(3+\frac{1}{x^2})}} = \frac{3-0}{\sqrt{3+0}}$$

■

Example 2.4.7. (Rationalization)
Evaluate

$$\lim_{x\to+\infty}(\sqrt{x+1}-\sqrt{x}).$$

"$\infty-\infty$" difference rule does not directly apply

Solution. (Recall the *Caveat* from last section!)

$a^2-b^2=(a+b)(a-b)$

$a=\sqrt{x+1}$ $b=\sqrt{x}$

$(x+1)-x=(\sqrt{x+1}+\sqrt{x})(\sqrt{x+1}-\sqrt{x})$

$$\lim_{x\to+\infty}(\sqrt{x+1}-\sqrt{x}) = \lim_{x\to+\infty}\frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{x\to+\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}}$$

$$= 0.$$

quotient rule

$$= \frac{\lim_{x\to\infty}1}{\lim_{x\to\infty}(\sqrt{x+1}+\sqrt{x})} = \frac{1}{\lim_{x\to\infty}\sqrt{x+1}+\lim_{x\to\infty}\sqrt{x}} = 0.$$

■

Exercise 2.4.1.

1. $\displaystyle\lim_{x\to-\infty}\frac{x^3+1}{-2x^3+x}=-\frac{1}{2}.$
2. $\displaystyle\lim_{x\to-\infty}\frac{x/x}{\sqrt{x^2+1}/x}=-1$ (Caution: $x<0\Rightarrow\frac{1}{x}=-\sqrt{\frac{1}{x^2}}$).
3. $\displaystyle\lim_{x\to+\infty}(\sqrt{x^2+x}-\sqrt{x^2-2})=\frac{1}{2}.$

$$\lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{1}{x^2}}} = -\frac{\lim_{x\to-\infty}1}{\lim_{x\to-\infty}\sqrt{1+\frac{1}{x^2}}} = -\frac{1}{\sqrt{\lim_{x\to-\infty}(1+\frac{1}{x^2})}} = \frac{-1}{\sqrt{1+0}} = -1$$

Example 2.4.8. $\displaystyle\lim_{x\to+\infty}\sin x=?$

2.5 Limits involving "e"

Definition 2.5.1.

$$e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x.$$

e is the base for natural log, $\log_e x = \ln x$. $= \log x$

$$e = 2.71828\ldots$$

x	-1000	-100	-10	10	100	1000
$\left(1+\frac{1}{x}\right)^x$	2.71964	2.73200	2.86797	2.59374	2.70481	2.71692

Remark. 1. Note that

$$e := \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x \neq \left(\lim_{x\to+\infty} 1+\frac{1}{x}\right)^x = 1!$$

$\underline{a^x}$

2. Motivation for defining e this wa will be clear later when we learn about differentiation.

$u^{ab} = (u^a)^b$

Example 2.5.1. Evaluate

$$\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x.$$

Solution.

$\left(1+\frac{1}{-x}\right)^x$

$$\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^{-y} = \lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^x = \lim_{x\to+\infty}\left[\left(1+\frac{1}{(-x)}\right)^{(-x)}\right]^{-1} \qquad (\text{ set } -x = y)$$

as $x \to +\infty$, $y \to -\infty$

$$= \left[\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y\right]^{-1}$$

$$= e^{-1}$$

$$= \lim_{y\to-\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^{-1} = \left[\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y\right]^{-1} = e^{-1}$$

■

Exercise 2.5.1. Evaluate $\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^{2x} = e^4$.

$\frac{1}{x/2}$

let $y = \frac{x}{2}$ as $x \to +\infty$, $y \to +\infty$

$4y = 2x$

$$= \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{4y} = \lim_{y\to+\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^4 = \left[\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y\right]^4$$

$$= e^4$$

MATH1520 University Mathematics for Applications **Fall 2021**

Chapter 3: Continuity

Learning Objectives:
(1) Explore the concept of continuity and examine the continuity of several functions.
(2) Investigate the intermediate value property.

3.1 Continuity

Eg. polynomial functions

Definition 3.1.1. A function f is **continuous** at $x = x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$. It means all three of these conditions are satisfied:

1. $f(x_0)$ is defined.
2. $\lim_{x \to x_0} f(x)$ exists.
3. They are equal.

If some of (1)-(3) are not satisfied, then $f(x)$ is discontinuous at x_0.

If $f(x)$ is continuous at every point in the domain, $f(x)$ is called a continuous function.

Informally, a function $f(x)$ is continuous at $x = x_0$ if the curve of $f(x)$ does not break up at x_0. A continuous function is one whose graph has no holes or gaps.

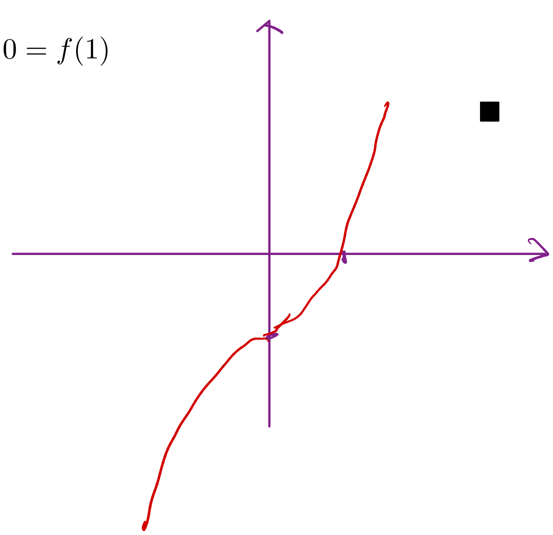
Example 3.1.1. Show that $f(x) = x^3 - 1$ is continuous at $x = 1$.

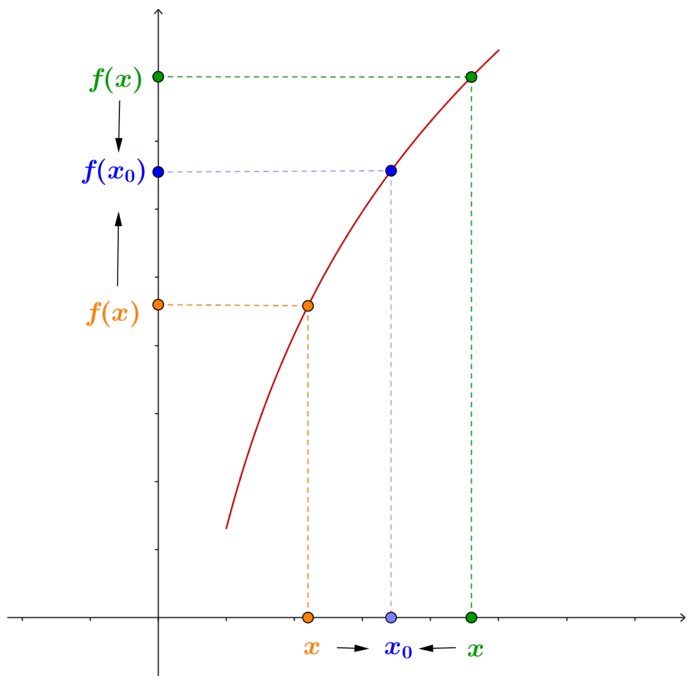
Solution.

$$f(1) = 0.$$

$$\lim_{x \to 1} f(x) = 1^3 - 1 = 0 = f(1)$$

(i.e., limit exists and is equal to $f(1)$.) ■





Example 3.1.2. Decide whether the function

$$f(x) = \begin{cases} x^3 - 1, & x \neq 1, \\ 1, & x = 1. \end{cases}$$

is continuous at $x = 1$.

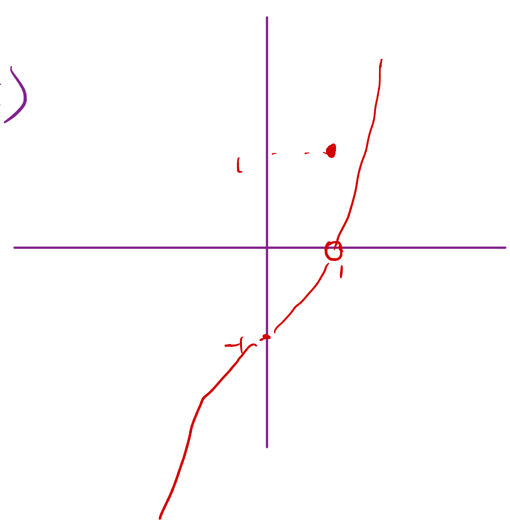
Solution. Since

$$\lim_{x \to 1} f(x) = 0 \neq f(1),$$

$f(x)$ is discontinuous at $x = 1$. ■

$f(1) = 1$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^3 - 1) = 0 \neq f(1)$

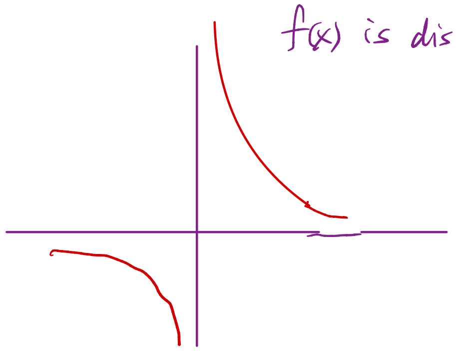


Example 3.1.3. Discuss the continuity of $f(x) = \dfrac{1}{x}$.

Solution. $f(x)$ is defined everywhere except at $x = 0$, and $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$ $\forall c \neq 0$ by the first propositions of Chapter 2. So $f(x)$ is continuous for all $x \neq 0$. ■

$f(c)$

$f(x)$ is discontinuous at $x=0$

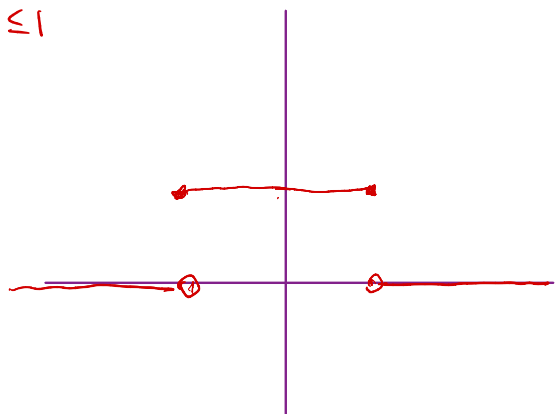


Example 3.1.4. Piecewise linear functions (e.g. step functions, the ceil/floor function, $f(x) = |x|$); piecewise continuous functions.

Eg. $f(x) = \begin{cases} 1 & \text{when } |x| \le 1 \Leftrightarrow -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

f is discontinuous at $x=1$

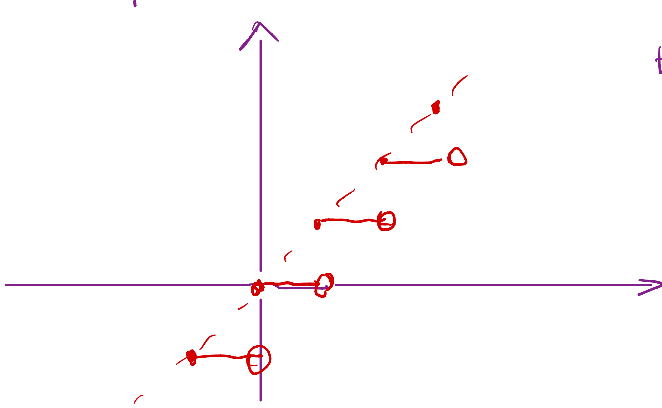
because $\lim_{x\to 1} f(x)$ does not exist



$$\left.\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} 1 = 1 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} 0 = 0 \end{aligned}\right\} \not\equiv$$

so f is discontinuous at $x=1$.

Eg. The floor function $f(x) = \lfloor x \rfloor$ is



this function is discontinuous at integer x.

continuous on $\mathbb{R} - \mathbb{Z}$

Proposition 3.1.1. **(Properties of continuity)**

1. Suppose $f(x)$ and $g(x)$ are continuous at $x = x_0$. It follows from Proposition 2 in Chapter 2 that:

   (a) $f(x) + g(x)$, $f(x) - g(x)$, $f(x)g(x)$ are continuous at $x = x_0$.

   (b) If $g(x_0) \neq 0$, then $\dfrac{f(x)}{g(x)}$ is continuous at $x = x_0$.

2. It follows from Proposition 3 in Chapter 2 that: If $g(x)$ is continuous at $x = x_0$ and $f(x)$ is continuous at $x = g(x_0)$. Then $(f \circ g)(x)$, i.e., $f(g(x))$ is continuous at $x = x_0$. In fact $\lim_{x \to x_0} f(g(x)) = \lim_{u \to g(x_0)} f(u) = f(g(x_0))$.

   $u = g(x)$

3. $x^a, a^x, \log_a x$ and trig functions are all continuous functions in the domain. As a consequence, their $+, -, \times, \div, \circ$ are all continuous in the domain.

Example 3.1.5.

1. If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to c} p(x) = p(c)$$

   and

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \text{ if } q(c) \neq 0.$$

   So a polynomial or a rational function is continuous wherever it is defined (i.e. $q(c) \neq 0$).

2. $f(x) = \dfrac{x-1}{x+1}$ is continuous at $x = 2$.

3. $f(x) = \dfrac{x^2-1}{x+1}$ is defined everywhere except at $x = -1$, so it is continuous everywhere except at $x \cancel{\neq} = -1$.

4. $g(x) = \ln\sqrt{x^2+1}$ is continuous on $\mathbb{R}$.

   regard as a composite function / change of variable

   let $u = \sqrt{x^2+1} = h(x)$ is continuous on $\mathbb{R}$

   $\ln u = \ln(h(x))$ is continuous on $\mathbb{R}$

   $\ln u$ is continuous on $\mathbb{R}^+$

   Range of $h(x) = [1, \infty) \subset \mathbb{R}^+$

Example 3.1.6. Discuss the continuity of the piecewise function:

$$f(x) = \begin{cases} x+1 & \text{if } x \leq 1, \\ 2x^2 & \text{if } x > 1. \end{cases}$$

Solution. For $x < 1$, $f(x) = x + 1$ is continuous on $(-\infty, 1)$;

For $x > 1$, $f(x) = 2x^2$ is continuous on $(1, +\infty)$;

At $x = 1$, $f(1) = 1 + 1 = 2$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+1) = 1 + 1 = 2.$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x^2 = 2 \cdot 1^2 = 2.$$

Because the left hand limit and the right hand limit exist and equal. So $\lim_{x\to 1} f(x) = 2 = f(1)$. Therefore $f(x)$ is continuous at all x. ∎

Example 3.1.7.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

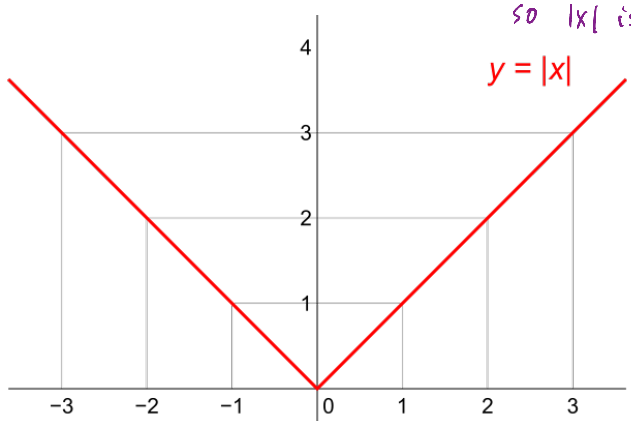
$\lim_{x\to 0^+}|x| = \lim_{x\to 0^+} x = 0$

||

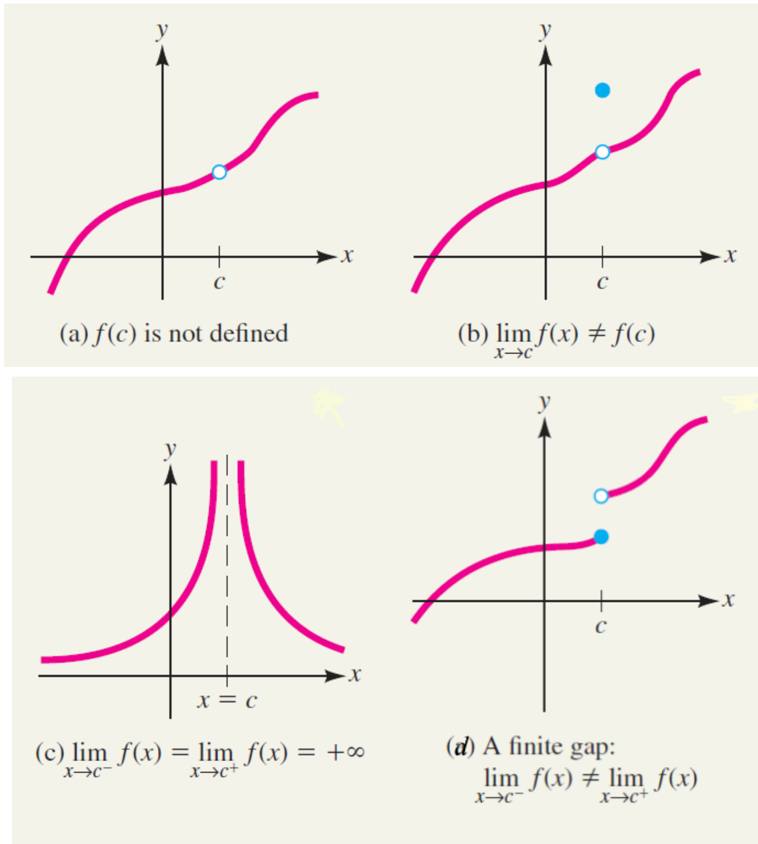
$\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = 0$

$|x|$ is a continuous everywhere and $\lim_{x\to 0}|x| = 0. = |0|$

so $|x|$ is continuous at $x=0$



Example 3.1.8. (Discontinuity)



(a) $f(c)$ is not defined

(b) $\lim_{x\to c} f(x) \neq f(c)$

(c) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = +\infty$

(*d*) A finite gap: $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

Example 3.1.9. For what value of A such that the following function is continuous at all x?

$$f(x) = \begin{cases} x^2 + x - 1 & \text{if } x \leq 0, \\ x + A & \text{if } x > 0. \end{cases}$$

Solution. Because $x^2 + x - 1$ and $x + A$ are polynomials, they are continuous everywhere except possibly at $x = 0$. Also $f(0) = 0^2 + 0 - 1 = -1$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2 + x - 1) = -1$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x + A) = A.$$

For $\lim_{x\to 0} f(x)$ to exist, the left hand limit and the right hand limit must be equal. So we must have $A = -1$. In which case

$$\lim_{x\to 0} f(x) = -1 = f(0).$$

This means that $f(x)$ is continuous for all x only when $A = -1$. ■

Proposition 3.1.2. $f(x)$ is continuous at $x = c$ if and only if

$$\lim_{h\to 0} f(c+h) = f(c).$$

$\lim_{x\to c} f(x) = f(c)$

$x = h + c$

Proof. Let $h = x - c$. Then $h \to 0$ as $x \to c$.

$$\lim_{x\to c} f(x) = \lim_{h\to 0} f(c+h).$$

□

Exercise 3.1.1.

1. Show that $\sqrt[3]{x^3+1}$ is a continuous function.
2. Show that $\left|\dfrac{x+1}{x-1}\right|$ is a continuous function on $\mathbb{R}\backslash\{1\}$.
3. Let
$$f(x) = \begin{cases} x^2 - 1, & x \le 0, \\ x + a, & x > 0. \end{cases}$$
Find a such that $f(x)$ is continuous at 0. (Ans: $a = -1$)

Example 3.1.10 (Using continuity to compute limits). $\lim_{x\to \infty^+} \sin\left(\frac{1}{x}\right) = ?$

$u = \frac{1}{x}$

$= \lim_{u\to 0^+} \sin u$ as $x \to +\infty$, $u \to 0^+$

$\sin x$ is continuous $\Downarrow$ $= \sin 0 = 0$

3.2 Continuity on $[a, b]$

Definition 3.2.1. Let $f : (a, b) \to \mathbb{R}$ be a function. Then f is said to be continuous on (a, b) if it is continuous at every point on (a, b). at every $x_0 \in (a,b)$ $\lim_{x\to x_0} f(x)$ computed by x in (a,b)

Next, let's assume $f : [a, b] \to \mathbb{R}$ be a function. What's the meaning of f being continuous at one of the end point a? $\lim_{x\to a} f(x)$ does not make sense because f is not defined on $x < a$. So to define the continuity at $x = a$, we only concern about the value $x > a$. Similarly, to discuss about the continuity at $x = b$, we only concern about the value $x < b$.

Definition 3.2.2. Let $f : [a, b] \to \mathbb{R}$ be a function.
Then f is said to be continuous at a if

$$\lim_{x\to a^+} f(x) = f(a).$$

f is said to be continuous at b if

$$\lim_{x\to b^-} f(x) = f(b).$$

Then f is said to be a continuous function on $[a, b]$ if f is continuous on $a \le x \le b$.

Example 3.2.1. Discuss the continuity of the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{x-1}{x} & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} \frac{x-1}{x} = 0$

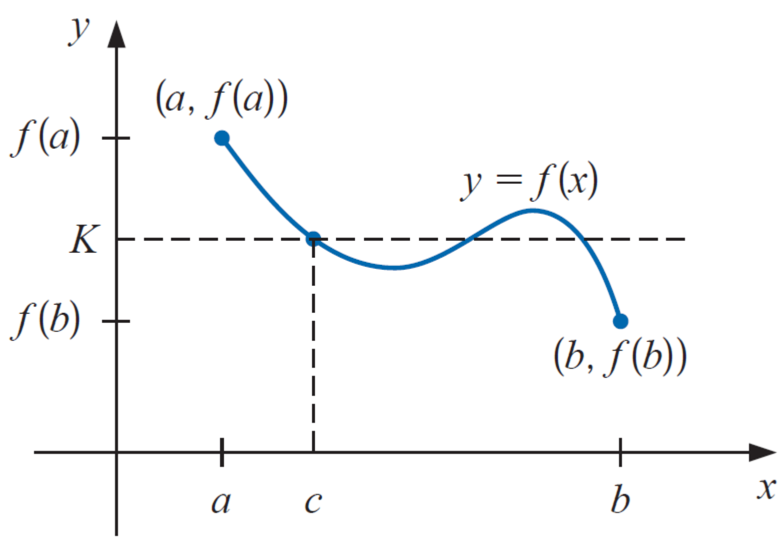
$f(1) = \frac{1-1}{1} = 0$

Solution. $f(x)$ is continuous on $(0, 1)$. $f(x)$ is also continuous at $x = 1$, but $\lim\limits_{x\to 0^+} f(x)$ does not exists. So f is not continuous at $x = 0$. ■

$f(0) = 0 \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x-1}{x} = -\infty \neq f(0)$

Theorem 3.2.1 (Intermediate Value Theorem or Intermediate Value Property). *Suppose f is a continuous function on $[a, b]$ and K is a number between $f(a)$ and $f(b)$. Then there exist a number c, between a and b, such that $f(c) = K$.*

Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.



Application: Root Finding

If $f(x)$ is continuous on $[a, b]$, $f(a)$ and $f(b)$ change sign, then, there exists at least one root of the function, that is, exists at least one $c \in (a, b)$, such that $f(c) = 0$.

Example 3.2.2. Show that $f(x) = x^5 - x + 1$ has a root.